# Mixed Tensor Products for Lie Superalgebras

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# Outline

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# The General Linear Lie Algebra

**Definition**

The general linear Lie algebra, denoted $\mathfrak{gl}(n)$, is the space of $n \times n$ matrices over $\mathbb{C}$, along with the following operation, called the Lie bracket:

$$[X, Y] = XY - YX,$$

where $XY$ and $YX$ denote matrix multiplication.

- Note the following identities:
  - $[X, X] = 0$ (alternating form)
  - $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$ (This is called the Jacobi identity.)
- We use $e_{ij}$ to denote the matrix with a 1 in the $(i, j)$-position and zeros everywhere else.

# More Lie Algebras

## Definition

The Lie algebra $\mathfrak{so}(3, \mathbb{R})$ ($\mathfrak{so}$: special orthogonal) is the space $\mathbb{R}^3$ of 3-dimensional vectors, where the Lie bracket of two vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$ is given by the cross product:

$$[\mathbf{x}, \mathbf{y}] = \mathbf{x} \times \mathbf{y}.$$

We have $[\mathbf{x}, \mathbf{x}] = \mathbf{x} \times \mathbf{x} = \mathbf{0}$, and the Jacobi identity for cross products:

$$[\mathbf{x}, [\mathbf{y}, \mathbf{z}]] + [\mathbf{y}, [\mathbf{z}, \mathbf{x}]] + [\mathbf{z}, [\mathbf{x}, \mathbf{y}]] = \mathbf{x} \times (\mathbf{y} \times \mathbf{z}) + \mathbf{y} \times (\mathbf{x} \times \mathbf{z}) + \mathbf{z} \times (\mathbf{x} \times \mathbf{y}) = 0.$$

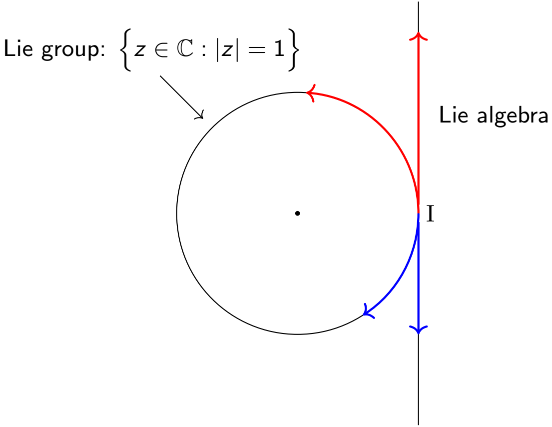
Thus $\mathfrak{so}(3, \mathbb{R})$ is a Lie algebra.

## Definition

Any vector space $V$ can be made into an abelian Lie algebra by $[x, y] = 0$ for all $x, y \in V$.

# Why Lie Algebras?

Lie algebras can be thought of as the "little sister" of Lie groups.




For each element of the Lie algebra, we can associate an element of the Lie group. This often lets us recover information about the group.

# Why Lie Algebras?, contd.

- Infinitesimal symmetry motions.
- Quantum mechanics and particle physics.
- Lie groups are symmetry groups for physical systems.
- Combinatorial applications of representation theory.

# Differential Operators

We work over the space of polynomials $\mathbb{C}[t_0, t_1, \ldots, t_n]$.

A differential operator is a combination of operators of the form $\frac{\partial}{\partial t_i}$ (abbrev. $\partial_i$) and $t_i$ (multiplication by $t_i$).

## Examples

The differential operator $\partial_0$ applied to the polynomial $2t_0^2t_1 + t_0t_2 + 3t_1$ yields

$$\frac{\partial}{\partial t_0}(2t_0^2t_1 + t_0t_2 + 3t_1) = 4t_0t_1 + t_2.$$

The differential operator $t_0$ applied to the same polynomial yields $2t_0^3t_1 + t_0^2t_2 + 3t_0t_1$. We can combine these operators to get the operator $t_0\partial_0$, which acts by first acting by $\partial_0$ and then by $t_0$. Applying that to our polynomial, we get $4t_0^2t_1 + t_0t_2$.

# Differential Operators, contd.

Now consider $\mathbb{C}[t_0^{\pm 1}, t_1, \ldots, t_n]$. Extend the definition of differential operators to include operators like $\frac{1}{t_0}$ (that is, multiplication by $t_0^{-1}$). We have

$$\frac{1}{t_0}(2t_0^2 t_1 + t_0 t_2 + 3t_1) = 2t_0 t_1 + t_2 + 3\frac{t_1}{t_0}.$$

**Definition**

The associative unital algebra $\mathcal{D}'(n)$ is the space of differential operators on $\mathbb{C}[t_0^{\pm 1}, t_1, \ldots, t_n]$ consisting of all sums of products of operators of the form $t_0\partial_i$ and $\frac{t_i}{t_0}$.

- $\mathcal{D}'(n)$ is in fact a Lie algebra by $[x, y] = xy - yx$.
- Since $\frac{t_i}{t_0} \cdot t_0\partial_j = t_i\partial_j$, we have $t_i\partial_j \in \mathcal{D}'(n)$ for all $1 \leq i, j \leq n$.
- $\mathfrak{gl}(n)$ is a subalgebra of $\mathcal{D}'(n)$: under the mapping $e_{ij} \mapsto t_i\partial_j$,

$$[e_{ij}, e_{kl}] \mapsto [t_i\partial_j, t_k\partial_l].$$

## The Universal Enveloping Algebra

Sometimes, we embed $\mathfrak{gl}(n)$ in its universal enveloping algebra $U(\mathfrak{gl}(n))$.

- It is an associative unital algebra, which means it has a multiplication somewhat like the one we are used to.
- However, as far as representation theory goes, $U(\mathfrak{gl}(n))$ and $\mathfrak{gl}(n)$ are essentially the same.

# Outline

# Tensor Products

Given algebras $A$ and $B$, we can form a new algebra $A \otimes B$ called the tensor product of $A$ and $B$.

- This is a way of "joining" two algebras.

The elements of $A \otimes B$ are linear combinations of elements of the form $a \otimes b$.

- "$\otimes$" is just a formal symbol.

Multiplication is given by $(a_1 \otimes b_1)(a_2 \otimes b_2) = (a_1 a_2) \otimes (b_1 b_2)$.

- Heuristically, the elements of $A$ and $B$ do not interact with each other.

# Mixed Tensor Products

In 2021, Grantcharov and Robitaille constructed a homomorphism

$$\varphi : U(\mathfrak{gl}(n+1)) \to \mathcal{D}'(n) \otimes U(\mathfrak{gl}(n)).$$

This is called a mixed tensor product.

Applications include

- Representation theory,
- It can be thought of as a form of Beilinson-Bernstein localization for parabolic subgroups (geometric interpretation),
- Invariant theory, and
- Algebraic combinatorics.

Grantcharov and Robitaille did many interesting things with this homomorphism, most notably an explicit computation of the image of the center of $U(\mathfrak{gl}(n+1))$ under $\varphi$.

**Note**: The center is the set of elements that commute with all other elements, and determines important representation theoretic information.

# Formulas for $\varphi$

- Let $\delta_{ab}$ denote the Kronecker delta function, which is 1 if $a = b$ and 0 otherwise:

$$\delta_{11} = \delta_{22} = 1 \quad \text{but} \quad \delta_{12} = \delta_{21} = 0.$$

- $R_1$ is a special element of $\mathcal{D}'(n) \otimes U(\mathfrak{gl}(n))$.

**Proposition (Grantcharov-Robitaille, 2021)**

*The following correspondence*

$$\begin{aligned} e_{ab} &\mapsto t_a\partial_b \otimes 1 + 1 \otimes e_{ab} + \delta_{ab}R_1, && (a, b \geq 1) \\ e_{a0} &\mapsto t_a\partial_0 \otimes 1 - \sum_{j=1}^{n} \frac{t_j}{t_0} \otimes e_{aj}, && (a \geq 1) \\ e_{0b} &\mapsto t_0\partial_b \otimes 1, && (b \geq 1) \\ e_{00} &\mapsto t_0\partial_0 \otimes 1 + R_1 \end{aligned}$$

*defines a homomorphism* $\varphi_R : U(\mathfrak{gl}(n+1)) \to \mathcal{D}'(n) \otimes U(\mathfrak{gl}(n))$.

# Outline

# Super vector spaces

**Definition**

A super vector space is a vector space $V = V_0 \oplus V_1$, where $V_0$ and $V_1$ are vector spaces. Elements of $V_0$ and $V_1$ are called homogeneous. For a nonzero homogeneous element $x$, we define a parity function $|\cdot|$ by

$$|x| = \begin{cases} 0 & \text{if } x \in V_0 \\ 1 & \text{if } x \in V_1. \end{cases}$$

If $|x| = 0$, we say $x$ is even and if $|x| = 1$, we say $x$ is odd.

- Every $x \in V$ can be written uniquely as $x = x_0 + x_1$, where $x_0 \in V_0$ is even and $x_1 \in V_1$ is odd.

# The Lie Superalgebra $\mathfrak{gl}(m|n)$

Consider the space of $(m+n)\times(m+n)$ matrices.

- Use $1, 2, \ldots, m, \bar{1}, \ldots, \bar{n}$ to index the rows and columns:

$$\begin{array}{c} \\ 1,\ldots,m \\ \bar{1},\ldots,\bar{n} \end{array} \overset{\begin{array}{cc} 1,\ldots,m & \bar{1},\ldots,\bar{n} \end{array}}{\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]} \quad \text{with} \quad \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \underbrace{\left[\begin{array}{c|c} A & 0 \\ \hline 0 & D \end{array}\right]}_{\text{even}} + \underbrace{\left[\begin{array}{c|c} 0 & B \\ \hline C & 0 \end{array}\right]}_{\text{odd}}.$$

- Parity function $|\cdot|$:

$$|1| = \cdots = |m| = 0 \quad \text{and} \quad |\bar{1}| = \cdots = |\bar{n}| = 1.$$

- Parity function $|\cdot|$ on elementary matrices $e_{ij}$ by $|e_{ij}| = |i| + |j|$.

# The Lie Superalgebra $\mathfrak{gl}(m|n)$, contd.

We can now define the Lie superbracket on $\mathfrak{gl}(m|n)$.

**Definition**

For homogeneous $X, Y \in \mathfrak{gl}(m|n)$, the Lie superbracket is defined by

$$[X, Y] = XY - (-1)^{|X| \cdot |Y|} YX.$$

- This matches the Lie bracket $XY - YX$ unless $X, Y$ are *both* odd.

# Applications of Lie Superalgebras

- Applications to physics, like Lie algebras.
- Supersymmetry: a theory that postulates a duality between bosons and fermions.
  - Postulates a host of new particles that have not been experimentally found yet.
- Symmetric tensor categories (STCs): Abstracts the properties of the set of representations Rep $G$ of a group $G$ into an object called a category.
- When trying to classify all STCs, Lie superalgebras (and supergroups) turn up.

# Supermathematics

- Supermathematics: a field of math created by adjoining “super” to everything.
- Bosons: integer spin; fermions: half-integer spin.
- Bosons are even, fermions are odd, and they “supercommute”.

| algebra | group | vector space | ring |
|---|---|---|---|
| ↓ | ↓ | ↓ | ↓ |
| superalgebra | supergroup | super vector space | superring |

Superman!

# Super Differential Operators

$\mathcal{F}$: set of sums of products of the variables $t_0^{\pm 1}, t_1, \ldots, t_m, t_{\bar{1}}, \ldots, t_{\bar{n}}$, along with

$$t_i t_j = (-1)^{|i||j|} t_j t_i.$$

## Definition

The associative unital algebra $\mathcal{D}'(m|n)$ is the space of differential operators on $\mathcal{F}$ consisting of all sums of products of operators of the form $t_0 \partial_i$ and $\frac{t_i}{t_0}$ for $i \in \{0, \ldots, m, \bar{1}, \ldots, \bar{n}\}$.

- Note: odd generators $t_{\bar{1}}, \ldots, t_{\bar{n}}$ form an "exterior algebra" $\Lambda_n$.

## Example

We have

$$(t_{\bar{1}}\partial_1 + t_1\partial_{\bar{1}})(t_1 t_{\bar{1}}) = (t_{\bar{1}}\partial_1)(t_1 t_{\bar{1}}) + (t_1\partial_{\bar{1}})(t_1 t_{\bar{1}}) = \underbrace{t_{\bar{1}}^2}_{=0} + t_1^2 = t_1^2.$$

# Outline

# Our Research

We generalized Grantcharov's and Robitaille's results to the superalgebra setting.

**Theorem (Erat-Kannan-Kanungo, 2024)**

*For any element $R$ in $Z(\mathcal{D}'(m|n) \otimes U(\mathfrak{gl}(m|n))) = \mathbb{C}[\mathcal{E}] \otimes Z(U(\mathfrak{gl}(m|n)))$, the correspondence given by*

$$\begin{aligned} e_{ab} &\mapsto t_a \partial_b \otimes 1 + 1 \otimes e_{ab} + \delta_{ab}(-1)^{|a||b|} R, && (a, b \in I) \\ e_{a0} &\mapsto t_a \partial_0 \otimes 1 - \sum_{j \in I} (-1)^{|a||j|} \frac{t_j}{t_0} \otimes e_{aj}, && (a \in I) \\ e_{0b} &\mapsto t_0 \partial_b \otimes 1, && (b \in I) \\ e_{00} &\mapsto t_0 \partial_0 \otimes 1 + R \end{aligned}$$

*defines a homomorphism $\varphi_R : U(\mathfrak{gl}(m+1|n)) \to \mathcal{D}'(m|n) \otimes U(\mathfrak{gl}(m|n))$.*

# Our Research, contd.

- We also generalized Grantcharov and Robitaille's results about the center of $U(\mathfrak{gl}(n+1))$.
- We found images of two different sets of generators of the center. These are called the Capelli Berezinian $C_{m|n}(T)$ and the Gelfand generators $G_k^{\mathfrak{gl}(m|n)}$.

**Theorem (Erat-Kannan-Kanungo, 2024)**

*We have*

$$\varphi(C_{m+1|n}(T+R)) = (\mathcal{E}-T)C_{m|n}(T+1).$$

*and*

$$\varphi(G_k^{\mathfrak{gl}(m+1|n)}) = \sum_{g=0}^{k-1}\binom{k}{g}R^g R_2^{k-1-g}(\mathcal{E}\otimes 1) + (m+1-n)R^k + \sum_{g=0}^{k-1}\binom{k}{g}R^g\left(1\otimes G_{k-g}^{\mathfrak{gl}(m|n)}\right)$$
$$-\sum_{s=2}^{k}\left(1\otimes G_{s-1}^{\mathfrak{gl}(m|n)}\right)\sum_{g=0}^{k-s}\binom{k}{g}R^g R_2^{k-s-g}.$$

We generalize a transition formula between these two generators—Newton's formula for $\mathfrak{gl}(n)$—to the superalgebra setting.

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# References


- S. Erat, A. S. Kannan, S. Kanungo (2024). "*Mixed Tensor Products, Capelli Berezinians, and Newton's Formula for* $\mathfrak{gl}(m|n)$." arXiv:2409.02422.
- D. Grantcharov and L. Robitaille (2021). "*Mixed tensor products and Capelli-type determinants*." arXiv:2102.07027. To appear in Proc. Amer. Math. Soc.

# Questions?

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# Thank You!

Thank You!